

Capacities on a Finite Lattice

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Abstract

In his influential work [1] Choquet systematically studied capacities on Boolean algebras in a topological space, and gave a probabilistic interpretation for completely monotone (and completely alternating) capacities. Beyond complete monotonicity we can view a capacity as a marginal condition for probability distribution over the distributive lattice of dual order ideals. In this paper we discuss a combinatorial approach when capacities are defined over a finite lattice, and investigate Fréchet bounds given the marginal condition, probabilistic interpretation of difference operators, and stochastic inequalities with completely monotone capacities.

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1 Introduction

Let L be a finite lattice with partial ordering \leq , and let $\hat{0}$ and $\hat{1}$ denote the minimum and the maximum element of L . A monotone function φ on L is called a *capacity* if $\varphi(\hat{0}) = 0$ and $\varphi(\hat{1}) = 1$. Let \mathcal{L} denote the collection of nonempty dual order ideals in L , and let \mathcal{X} be an \mathcal{L} -valued random variable on some probability space (Ω, \mathbb{P}) , distributed as $\mathbb{P}(\mathcal{X} = V) = f(V)$. Assuming $\mathbb{P}(\hat{0} \in \mathcal{X}) = 0$, we can construct a capacity φ by

$$(1.1) \quad \varphi(x) = \mathbb{P}(x \in \mathcal{X}), \quad x \in L.$$

From another viewpoint, the collection of capacities on L is a convex polytope, any element of which can be represented as the convex combination

$$(1.2) \quad \varphi(x) = \sum_{V \in \mathcal{L}} f(V) \chi_V(x), \quad x \in L,$$

where χ_V denotes an indicator function on V . In the way of formulating (1.2), the weight $f(V)$ is viewed as a *probability mass function* (pmf) for \mathcal{X} , by which (1.2) is deemed to be (1.1). This probabilistic interpretation of capacity was first considered by Choquet [1] and independently by Murofushi and Sugeno [6]. It should be noted, however, that the choice of f is not necessarily unique (see Examples 3.3 and 3.4).

Let X be an L -valued random variable, distributed as $\mathbb{P}(X = x) = f(x)$. If $f(\hat{0}) = 0$ then the *cumulative distribution function* (cdf)

$$(1.3) \quad \varphi(x) = \sum_{y \leq x} f(y), \quad x \in L,$$

becomes a capacity, also known as a *belief function* in [2]. The function f in (1.3) is called the *Möbius inverse* of φ .

For $a_1, a_2, \dots \in L$, we define the *difference operator* ∇_{a_1} by

$$(1.4) \quad \nabla_{a_1} \varphi(x) = \varphi(x) - \varphi(x \wedge a_1), \quad x \in L,$$

and the *successive difference operator* ∇_{a_1, \dots, a_n} recursively by

$$(1.5) \quad \nabla_{a_1, \dots, a_n} \varphi = \nabla_{a_n} (\nabla_{a_1, \dots, a_{n-1}} \varphi), \quad n = 2, 3, \dots$$

Then the monotonicity of φ is characterized by $\nabla_a \varphi \geq 0$ for any $a \in L$. Moreover, if $\nabla_{a_1, \dots, a_n} \varphi \geq 0$ for any $a_1, \dots, a_n \in L$ and for any $n \geq 1$ then φ is called *completely monotone* (or monotone of order ∞ ; see [1]). The complete monotonicity of φ is necessary and sufficient for the existence of a (necessarily unique) pmf satisfying (1.3). This crucial observation was made by Choquet [1] for the class of compact sets in a topological space, and it is now known as the Choquet theorem which has been instrumental in the studies of random sets. See [5] for a comprehensive review on random sets on topological spaces. This result in case of lattices was due to Norberg [7] who studied measures on continuous posets.

By equipping \mathcal{L} with the order relation $U \preceq V$ by $U \supseteq V$, we obtain the distributive lattice \mathcal{L} which embeds L as the subposet $\mathcal{L}_0 := \{\langle a \rangle^* : a \in L\}$ of principal dual order ideals. Then we can introduce a completely monotone capacity Φ on \mathcal{L} , and call it a *completely monotone extension* of φ if it satisfies the marginal condition

$$(1.6) \quad \varphi(x) = \Phi(\langle x \rangle^*), \quad x \in L.$$

The marginal condition (1.6) is equivalent to (1.2), and the pmf $f(V)$ can be obtained from the Möbius inversion of Φ . By the same token, (1.1) is the marginal condition (1.6) when $\Phi(U) = \mathbb{P}(\mathcal{X} \preceq U)$ is a cdf for \mathcal{X} .

In Section 2 we investigate the properties of the Möbius inversion by which the successive difference operators are fully characterized. Particularly we can show the Choquet theorem for a finite lattice. Consequently, we can represent the successive difference operator

$$(1.7) \quad \nabla_{a_1, \dots, a_n} \varphi(x) = \mathbb{P}(X \leq x, X \not\leq a_i \text{ for all } i = 1, \dots, n)$$

when φ is completely monotone.

In Section 3 we consider the optimal bounds for $\Phi(U)$, called *Fréchet bounds*, subject to the marginal condition (1.6). We present a combinatorial approach to the Fréchet bounds, and formulate the optimal lower bound $\lambda(\varphi; a, b)$ for $\Phi(\langle a, b \rangle^*)$ at the dual order ideal $\langle a, b \rangle^*$ generated by a pair $\{a, b\}$ of L . We can introduce a difference operator by replacing $\varphi(a \wedge x)$ with $\lambda(\varphi; a, x)$ in (1.4), and call it “ λ -difference,” denoted by $\Lambda_{a_1} \varphi$. The resulting successive λ -difference operator $\Lambda_{a_1, \dots, a_n} \varphi$ parallels the characterization of $\nabla_{a_1, \dots, a_n} \varphi$ via (1.7). In Section 3.2 we can show that there exists an \mathcal{L} -valued random variable \mathcal{X} satisfying

$$(1.8) \quad \Lambda_{a_1, \dots, a_n} \varphi(x) = \mathbb{P}(x \in \mathcal{X}, a_i \notin \mathcal{X} \text{ for all } i = 1, \dots, n)$$

given the marginal condition (1.1).

In Section 4 we briefly discuss completely alternating capacities and their probabilistic interpretation in terms of dual capacities. Then we investigate a stochastic comparison between $\varphi(x) = \mathbb{P}(x \in \mathcal{X})$ and $\psi(y) = \mathbb{P}(Y \leq y)$, and obtain a sufficient condition for $\mathbb{P}(Y \in \mathcal{X}) = 1$, which is characterized by the two types of difference operator introduced earlier.

Our notation of set operations is fairly standard. The set difference $A \setminus B$ is defined by $\{x \in A : x \notin B\}$, and the inclusion relation $A \subset B$ means that A is a strictly smaller subset of B .

2 Successive difference functionals

By $R(L)$ we denote the space of real-valued functions on L . In this section we consider (1.5) defined over $\varphi \in R(L)$. The operator ∇_{a_1, \dots, a_n} does not depend on the order of a_i ’s. It is also easy to see that $\nabla_{a_1, \dots, a_n} \varphi(x) = 0$ if $x \leq a_i$ for some $i \leq n$; in particular, if $a_n \leq a_i$ for some $i \leq n-1$ then $\nabla_{a_1, \dots, a_n} \varphi(x) = \nabla_{a_1, \dots, a_{n-1}} \varphi(x) - \nabla_{a_1, \dots, a_{n-1}} \varphi(x \wedge a_n) = \nabla_{a_1, \dots, a_{n-1}} \varphi(x)$. Thus, we can introduce the *successive difference functional* $\nabla_A^b \varphi = \nabla_{a_1, \dots, a_n} \varphi(b)$ for a nonempty subset

$A = \{a_1, \dots, a_n\}$ of L and $b \in L$. We can expand it to

$$(2.1) \quad \nabla_A^b \varphi = \sum_{A' \subseteq A} (-1)^{|A'|} \varphi(\bigwedge A' \wedge b), \quad \varphi \in R(L),$$

where

$$\bigwedge A' = \begin{cases} \hat{1} & \text{if } A' = \emptyset; \\ \bigwedge_{a \in A'} a & \text{if } A' \neq \emptyset, \end{cases}$$

denotes the greatest lower bound of a subset A' of L . The Möbius inverse f in (1.3) is uniquely determined by

$$(2.2) \quad f(x) = \sum_{y \leq x} \varphi(y) \mu(y, x),$$

where μ is called the *Möbius function*.

Here we denote the half-open interval $\{x \in L : a \leq x < b\}$ by $[a, b)$. We say “ b covers a ” if $a < b$ and there is no other element between a and b (i.e., $[a, b) = \{a\}$), and “ A' dominates A ” if $A' \subseteq A$ and for any $x \in A$ there exists some $y \in A'$ satisfying $x \leq y$. It is easy to see that $\nabla_A^b = \nabla_{A'}^b$ if A' is a dominating subset of A .

The Möbius function over the lattice L can be constructed via the “cross-cut” property of Lemma 2.1.

Lemma 2.1 (Corollary 3.9.4 of Stanley [10]). *Let $a < b$, and let $C \subseteq [a, b)$. If C dominates $[a, b)$ then the Möbius function satisfies*

$$\mu(a, b) = \sum_{k=1}^{|C|} (-1)^k N_k$$

where $|C|$ denotes the number of elements in C , and N_k is the number of k -element subsets C' of C satisfying $\bigwedge C' = a$.

A nonempty subset of a poset is called *antichain* if any two distinct elements of the subset are incomparable; a singleton $\{a\}$ is a trivial antichain. Let $b \in L$ be fixed. An n -element subset $A = \{a_1, \dots, a_n\}$ of L is said to be a *b -meet antichain* if $\{a_1 \wedge b, \dots, a_n \wedge b\}$ is an n -element antichain. We call a singleton $\{a\}$ a trivial b -meet antichain only when $b \not\leq a$.

By $L_A^b := \{\bigwedge A' \wedge b : A' \subseteq A\}$ we denote the induced subposet of L . Then L_A^b is a lattice with the minimum $\bigwedge A \wedge b$, and shares the same meet \wedge with L . If $A = \{a_1, \dots, a_n\}$ is a b -meet antichain, then the maximum b of L_A^b covers exactly n elements $a_1 \wedge b, \dots, a_n \wedge b$.

Lemma 2.2. *Let A be a b -meet antichain, and let μ_A^b be the Möbius function of the lattice L_A^b . Then*

$$\nabla_A^b \varphi = \sum_{x \in L_A^b} \varphi(x) \mu_A^b(x, b).$$

Proof. Let $x \in L_A^b \setminus \{b\}$ be fixed, and let $C_x = \{a \wedge b : x \leq a \wedge b, a \in A\}$ be a dominating subset of $\{z \in L_A^b : x \leq z < b\}$. By Lemma 2.1 we obtain

$$\mu_A^b(x, b) = \sum_{k=1}^{|C_x|} (-1)^k N_k = \sum_{A' \subseteq A} (-1)^{|A'|} \chi_{\{\bigwedge A' \wedge b = x\}},$$

where

$$\chi_{\{\dots\}} = \begin{cases} 1 & \text{if } \{\dots\} \text{ is true;} \\ 0 & \text{if } \{\dots\} \text{ is false,} \end{cases}$$

is the indicator function for the statement $\{\dots\}$. Note that the right-hand expression of summation also produces the value $\mu(b, b) = 1$ when $x = b$. Thus, we obtain

$$\sum_{x \in L_A^b} \varphi(x) \mu_A^b(x, b) = \sum_{x \in L_A^b} \varphi(x) \sum_{A' \subseteq A} (-1)^{|A'|} \chi_{\{\bigwedge A' \wedge b = x\}},$$

which is equal to (2.1). \square

For the next lemma we assume that $b \not\leq a$ for any $a \in A$. Then we can find a subset $\tilde{A} \subseteq A$ such that (i) \tilde{A} is a b -meet antichain and (ii) $a \in A$ implies $a \wedge b \leq a' \wedge b$ for some $a' \in \tilde{A}$, and call it a “maximal b -meet antichain” of A . And we can reduce ∇_A^b to $\nabla_{\tilde{A}}^b$.

Lemma 2.3. *If \tilde{A} is a maximal b -meet antichain of A then $\nabla_A^b = \nabla_{\tilde{A}}^b$.*

Proof. Assume $\tilde{A} \subset A$. Let $a \in A \setminus \tilde{A}$ and $\tilde{a} \in \tilde{A}$ be such that $a \wedge b \leq \tilde{a} \wedge b$. Then we set $A' = A \setminus \{a\}$ and $A'' = A' \setminus \{\tilde{a}\}$, and obtain

$$\nabla_A^b = \nabla_{A''}^b - \nabla_{A''}^{b \wedge \tilde{a}} - \nabla_{A''}^{b \wedge a} + \nabla_{A''}^{b \wedge \tilde{a} \wedge a} = \nabla_{A'}^b.$$

We repeat further reduction, if necessary, until $A' = \tilde{A}$. \square

Theorem 2.4 verifies (1.7) when the Möbius inverse f represents the pmf for an L -valued random variable X .

Theorem 2.4. *The Möbius inverse f of φ satisfies*

$$(2.3) \quad \nabla_A^b \varphi = \sum_{x \in \pi_A^b} f(x)$$

where

$$\pi_A^b = \{x \in L : x \leq b, x \not\leq a \text{ for all } a \in A\}.$$

Proof. If $b \leq a$ for some $a \in A$ then $\pi_A^b = \emptyset$, for which we stipulate that the summation in (2.3) vanishes. Otherwise, we can find a maximal b -meet antichain \tilde{A} of A . It is easily observed that $\pi_{\tilde{A}}^b = \pi_A^b$; thus, it suffices to show (2.3) for \tilde{A} by Lemma 2.3. Henceforth, we assume that A is a b -meet antichain.

Here we can define the function \tilde{f} on the lattice L_A^b by setting

$$\tilde{f}(a) = \sum_{x \in \pi^a} f(x), \quad a \in L_A^b,$$

where $\pi^a = \{x \in L : x \leq a, x \not\leq a' \text{ whenever } a' < a \text{ in } L_A^b\}$. Since $\{\pi^a\}_{a \in L_A^b}$ partitions L , we obtain

$$\varphi(a) = \sum_{a' \leq a \text{ in } L_A^b} \tilde{f}(a'), \quad a \in L_A^b,$$

which implies that \tilde{f} is the Möbius inverse of φ over L_A^b . In particular, we can show that $\tilde{f}(b) = \nabla_A^b \varphi$ by Lemma 2.2. Note that $\pi^b = \pi_A^b$. Therefore, $\tilde{f}(b)$ is equal to the right-hand side of (2.3). \square

The following result is the immediate corollary which implies the Choquet theorem for capacities on a finite lattice.

Corollary 2.5. *Assume $\varphi(\hat{0}) \geq 0$. The Möbius inverse f of φ is nonnegative if and only if φ is completely monotone.*

Proof. Theorem 2.4 clearly implies the necessity of complete monotonicity. Note that $f(\hat{0}) = \varphi(\hat{0}) \geq 0$. For any $b > \hat{0}$ we can choose the collection A of all the elements covered by b , and obtain $\pi_A^b = \{b\}$ and $\nabla_A^b \varphi = f(b)$ in Theorem 2.4. Thus, the complete monotonicity of φ is also sufficient. \square

A subset V of L is called an *order ideal* (or a *down-set*) if $x \leq y$ and $y \in V$ imply $x \in V$. By $\langle A \rangle$ we denote the order ideal $\{x \in L : x \leq a \text{ for some } a \in A\}$ generated by a subset A of L . Then there is the one-to-one correspondence between antichains A and nonempty order ideals V via $V = \langle A \rangle$ (cf. [10]). Furthermore, we have $\nabla_A^b \equiv \nabla_V^b$ since A dominates $V = \langle A \rangle$.

Proposition 2.6. *Suppose that V is an order ideal of L . Then the Möbius inverse f of φ has the support $\{x \in L : f(x) \neq 0\}$ on V if and only if $\nabla_V^b \varphi = 0$ for every $b \notin V$.*

Proof. Let A be the antichain corresponding to V satisfying $V = \langle A \rangle$, and let \tilde{L} be the subposet of L induced on the subset $L \setminus V$. Then we can define the function $\tilde{\varphi}$ on \tilde{L} by setting

$$\tilde{\varphi}(b) = \sum_{x \leq b \text{ in } \tilde{L}} f(x), \quad b \in \tilde{L}.$$

By restricting f on \tilde{L} , we can view f as the Möbius inverse of $\tilde{\varphi}$. By introducing the subset π_A^b from Theorem 2.4, we can find that

$$\tilde{\varphi}(b) = \sum_{x \in \pi_A^b} f(x) = \nabla_A^b \varphi = \nabla_V^b \varphi.$$

Hence, $f \equiv 0$ on \tilde{L} if and only if $\nabla_V^b \varphi = 0$ for all $b \in \tilde{L}$. \square

3 Completely monotone extensions

A subset U of L is called a *dual order ideal* (or an *up-set*) if $x \in U$ and $x \leq y$ imply $y \in U$. By $\langle A \rangle^*$ we denote the up-set $\{x \in L : x \geq a \text{ for some } a \in A\}$ generated by a subset A of L ; thus, setting the one-to-one correspondence between antichains A and nonempty dual order ideals U via $U = \langle A \rangle^*$. We write simply $\langle a_1, \dots, a_n \rangle^*$ if $A = \{a_1, \dots, a_n\}$ is explicitly specified, and particularly we call it *principal* when the up-set $\langle a \rangle^*$ is generated by a singleton $\{a\}$. The collection $\mathcal{J}^*(L)$ of dual order ideals of L is a distributive lattice ordered by inclusion (cf. [10]), and so is the subposet of $\mathcal{J}^*(L)$ induced on the set of nonempty dual order ideals, denoted by \mathcal{L} . The poset \mathcal{L} is poset-isomorphic to the distributive lattice of dual order ideals on the subposet $L \setminus \{\hat{1}\}$. In what follows we assume that \mathcal{L} is equipped with the reverse inclusion relation \preceq so that $U \preceq V$ if $U \supseteq V$.

Example 3.1. Let $L = \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$ be a three-element Boolean lattice ordered by inclusion, where we express the subset $\{1, 2\}$ simply by “12.” Then the distributive lattice

$$\mathcal{L} = \{\langle \emptyset \rangle^*, \langle 1, 2, 3 \rangle^*, \langle 1, 2 \rangle^*, \langle 1, 3 \rangle^*, \langle 2, 3 \rangle^*, \langle 1, 23 \rangle^*, \langle 2, 13 \rangle^*, \langle 3, 12 \rangle^*, \langle 1 \rangle^*, \langle 2 \rangle^*, \langle 3 \rangle^*, \\ \langle 12, 13, 23 \rangle^*, \langle 12, 13 \rangle^*, \langle 12, 23 \rangle^*, \langle 13, 23 \rangle^*, \langle 12 \rangle^*, \langle 13 \rangle^*, \langle 23 \rangle^*, \langle 123 \rangle^*\}$$

has the minimum $\langle \emptyset \rangle^*$ and the maximum $\langle 123 \rangle^*$.

By $M_1(L)$ we denote the collection of nonnegative monotone functions on L , and by $M_\infty(\mathcal{L})$ the collection of nonnegative completely monotone functions on \mathcal{L} . As L is poset-isomorphic to the subposet \mathcal{L}_0 of \mathcal{L} induced on the set of principal dual order ideals, there is a natural projection $\Pi(\Phi) = \varphi$ via (1.6) from $\Phi \in M_\infty(\mathcal{L})$ to $\varphi \in M_1(L)$. The map Π is surjective, but not bijective unless L is linearly ordered. Proposition 3.2 is given by Murofushi and Sugeno [6] who demonstrated a construction of (1.2) by applying a “greedy method.”

Proposition 3.2. *The map Π is surjective from $M_\infty(\mathcal{L})$ onto $M_1(L)$.*

Proof. If $\varphi \equiv 0$ then $\Phi \equiv 0$ obviously satisfies $\Phi(\Phi) = \varphi$. Assume $\varphi \in M_1(L)$ with $\varphi(\hat{1}) > 0$. Then we can consider the map $U(t) = \{a \in L : \varphi(a) > t\}$ from $[0, \varphi(\hat{1}))$ to $\mathcal{J}^*(L)$. It is a step-wise decreasing map $U(t) \equiv U(r_{i-1})$ for $t \in [r_{i-1}, r_i)$ with $0 = r_0 < r_1 < \dots < r_m = \varphi(\hat{1})$. Then we can assign $f(V) = r_i - r_{i-1} > 0$ if $V = U(r_{i-1})$ for some i ; otherwise, $f(V) = 0$. Clearly the marginal condition (1.2) holds, and

$$(3.1) \quad \Phi(U) = \sum_{V \preceq U} f(V)$$

determines $\Phi \in M_\infty(\mathcal{L})$ as desired. \square

Example 3.3. Let L be the Boolean lattice from Example 3.1. Then

$$\varphi_c(x) = \begin{cases} 1 & \text{if } x = 123; \\ c & \text{if } x = 12, 13, \text{ or } 23; \\ 0 & \text{otherwise,} \end{cases}$$

is a capacity on L if $0 \leq c \leq 1$. By the greedy method we can construct a completely monotone extension

$$\Phi_c(U) = \begin{cases} 1 & \text{if } U = \langle 123 \rangle^*; \\ c & \text{if } \langle 12, 13, 23 \rangle^* \preceq U \prec \langle 123 \rangle^*; \\ 0 & \text{otherwise.} \end{cases}$$

If φ is completely monotone then the Möbius inverse f of φ can induce the Möbius extension Φ via (3.1) by setting $f(\langle x \rangle^*) = f(a)$ for $x \in L$ and $f \equiv 0$ on $\mathcal{L} \setminus \mathcal{L}_0$. The converse is also true: If the Möbius inverse f of a completely monotone extension Φ of φ has the support $\{U \in \mathcal{L} : f(U) \neq 0\}$ in \mathcal{L}_0 then φ is completely monotone, uniquely formulated by (1.3) with $f(x) = f(\langle x \rangle^*)$.

Example 3.4. In Example 3.3 we can find $\varphi_{1/3} \in M_\infty(L)$. Then the Möbius inverse

$$f(V) = \begin{cases} 1/3 & \text{if } V = \langle 12 \rangle^*, \langle 13 \rangle^*, \text{ or } \langle 23 \rangle^*; \\ 0 & \text{otherwise,} \end{cases}$$

determines the Möbius extension Φ of $\varphi_{1/3}$.

The Möbius extension can be characterized by its values at dual order ideals of the form $\langle a, b \rangle^*$.

Proposition 3.5. Φ is the Möbius extension of φ if and only if

$$(3.2) \quad \Phi(\langle a, b \rangle^*) = \varphi(a \wedge b) \text{ for every pair } \{a, b\}.$$

Proof. Let f be the Möbius inverse of Φ . Then we can observe that

$$\Phi(\langle a, b \rangle^*) = \varphi(a \wedge b) + \sum \{f(U) : a, b \in U \text{ and } a \wedge b \notin U\}.$$

Hence, Φ is the Möbius extension of φ and f is supported by \mathcal{L}_0 if and only if it satisfies (3.2). \square

3.1 Fréchet bounds

Kellerer [4] and Rüschendorf [9] investigated the optimal bounds analogous to the classical Fréchet bounds systematically for various marginal problems. Let $R(\mathcal{L})$ be the space of real-valued functions on \mathcal{L} . Given $\Phi \in M_\infty(\mathcal{L})$ we can formulate the nonnegative linear functional

$$\Phi(g) = \sum_{V \in \mathcal{L}} f(V)g(V), \quad g \in R(\mathcal{L}),$$

where f is the Möbius inverse of Φ . Assuming $\varphi \in M_1(L)$, we can define the Fréchet bound

$$(3.3) \quad B_\varphi(g) = \min\{\Phi(g) : \Pi(\Phi) = \varphi\}$$

for any $g \in R(\mathcal{L})$. Duality follows from the relationship between primal and dual problem of linear programming, but it is also viewed as a straightforward application of the Hahn-Banach theorem (cf. Kellerer [4]).

Theorem 3.6. *The dual problem*

$$(3.4) \quad S^\varphi(g) = \max \left\{ \sum_{x \in L} r_x \varphi(x) : \sum_{x \in V} r_x \leq g(V), V \in \mathcal{L} \right\}.$$

satisfies $B_\varphi(g) = S^\varphi(g)$ for any $g \in R(\mathcal{L})$.

Proof. We can introduce a function of the form

$$(3.5) \quad r(V) = \sum_{x \in L} r_x \chi_{\{x \in V\}}, \quad V \in \mathcal{L}$$

so that the inequality constraints in (3.4) are simply stated as $r \leq g$. Suppose that $\Phi_0 \in \Pi^{-1}(\varphi)$ attains $B_\varphi(g)$, and that r_0 of the form (3.5) satisfies $r_0 \leq g$ and attains $S^\varphi(g)$. Then we obtain $S^\varphi(g) = \Phi_0(r_0) \leq \Phi_0(g) = B_\varphi(g)$. Thus, $S^\varphi(g)$ is a lower bound for $B_\varphi(g)$, and the equality holds if g is in a form of (3.5).

Now let $g \in R(\mathcal{L})$ be fixed. Since S^φ is sublinear, satisfying $S^\varphi(g_1 + g_2) \geq S^\varphi(g_1) + S^\varphi(g_2)$, by the Hahn-Banach theorem we can find a linear functional Ψ such that $S^\varphi(h) \leq \Psi(h)$ for any $h \in R(\mathcal{L})$, in which the equality holds if h is in the form of (3.5) or $h = g$. Then Ψ is a nonnegative linear functional corresponding to $\Psi \in M_\infty(\mathcal{L})$, and it satisfies $\Pi(\Psi) = \varphi$. Hence, we have shown that $B_\varphi(g) \leq \Psi(g) = S^\varphi(g)$, which completes the proof. \square

Let $U \in \mathcal{L}$, and let $g_U(V) = \chi_{\{V \preceq U\}}$. Then we have $\Phi(U) = \Phi(g_U)$, and accordingly we simply write $B_\varphi(U)$ for $B_\varphi(g_U)$ in (3.3). In the rest of this subsection we investigate the Fréchet bound $B_\varphi(U)$.

Proposition 3.7. *If $\varphi \in M_\infty(L)$ then $B_\varphi(U)$ is the Möbius extension of φ .*

Proof. For each $U \in \mathcal{L}$, we can express $U = \langle A \rangle^*$ with antichain A , and observe that $\varphi(\bigwedge A) \leq B_\varphi(\langle A \rangle^*)$. Let Φ be the Möbius extension of φ . Then we can find $\Phi(\langle A \rangle^*) = \varphi(\bigwedge A)$, and therefore, $\Phi(\langle A \rangle^*) = B_\varphi(\langle A \rangle^*)$. \square

Example 3.8. In general, the Fréchet bound $B_\varphi(U)$ may not be a completely monotone extension of φ . Continuing from Example 3.3, we can find that

$$B_{\varphi_{2/3}}(U) = \begin{cases} 1 & \text{if } U = \langle 123 \rangle^*; \\ 2/3 & \text{if } U = \langle 12 \rangle^*, \langle 13 \rangle^*, \text{ or } \langle 23 \rangle^*; \\ 1/3 & \text{if } U = \langle 12, 13 \rangle^*, \langle 12, 23 \rangle^*, \text{ or } \langle 13, 23 \rangle^*; \\ 0 & \text{otherwise,} \end{cases}$$

is a completely monotone extension of $\varphi_{2/3}$ even though $\varphi_{2/3} \notin M_\infty(L)$. Whereas,

$$B_{\varphi_{1/2}}(U) = \begin{cases} 1 & \text{if } U = \langle 123 \rangle^*; \\ 1/2 & \text{if } U = \langle 12 \rangle^*, \langle 13 \rangle^*, \text{ or } \langle 23 \rangle^*; \\ 0 & \text{otherwise,} \end{cases}$$

is not completely monotone.

By \mathcal{T} we denote the class of connected acyclic graphs (i.e., trees) with vertex set on L . The vertex set of a tree G is also denoted by G , and the edge set $E(G)$ is a collection of pairs $\{a, b\}$ in G . Then we can associate a tree G with φ by setting

$$\varphi(G) = \sum_{a \in G} \varphi(a) - \sum_{\{a, b\} \in E(G)} \varphi(a \vee b).$$

Let $a \in G$ be fixed. Then we can introduce the unique rooted tree on G as follows: For $x, y \in G$, x is a descendant of y (and y is an ancestor of x) if the

path from x to a in G contains the path from y to a , and a becomes the root of the tree. The rooted tree is a directed graph (digraph) in which the ordered pair (x, y) represents the edge with y being the parent of x (i.e., the immediate ancestor of x). By $E(G; a)$ we denote the edge set of the rooted tree with the root a . By defining

$$\varphi(G; a) = \sum_{(x,y) \in E(G;a)} [\varphi(x \vee y) - \varphi(x)],$$

we can formulate $\varphi(G)$ equivalently by

$$(3.6) \quad \varphi(G) = \varphi(a) - \varphi(G; a).$$

Observe that $\varphi(G; a) \geq 0$, and therefore, that $\varphi(G) \leq \varphi(a)$. Moreover, we can obtain the following result as an immediate application of Theorem 3.6.

Lemma 3.9. $\varphi(G) \leq B_\varphi(\langle G \rangle^*)$ for any $G \in \mathcal{T}$.

In the proof of Lemma 3.9 it is convenient to define a graph restricted on a down-set: For a tree $G \in \mathcal{T}$ and a down-set D , we will define the subgraph $G|_D$ by setting $G|_D := G \cap D$ and $E(G|_D) := \{\{a, b\} \in E(G) : a \vee b \in D\}$.

Proof of Lemma 3.9. Let $g(V) = \chi_{\{V \preceq \langle G \rangle^*\}}$ and

$$r(V) = \sum_{a \in G} \chi_{\{a \in V\}} - \sum_{\{a,b\} \in E(G)} \chi_{\{a \vee b \in V\}}$$

for $V \in \mathcal{L}$. Note that r is in the form of (3.5). Since $|G| = |E(G)| + 1$, we can observe that $r(V) = g(V) = 1$ if $V \preceq \langle G \rangle^*$. Suppose that $V \not\preceq \langle G \rangle^*$. Then the down-set $D = L \setminus V$ contains at least one vertex of G . If the subgraph $G|_D$ has k connected components, we can find that $r(V) = 1 - k \leq 0$. Thus, we obtain $r \leq g$, and therefore, $\varphi(G) \leq S^\varphi(g)$. The proof is complete by Theorem 3.6. \square

In what follows we say “a path H from a to b ,” or simply write $H = (a, \dots, b)$ when $H \in \mathcal{T}$ and a and b are the only leaves in H (i.e., the two opposite ends of the path). By Lemma 3.9 we have $\varphi(H) \leq B_\varphi(\langle H \rangle^*) \leq B_\varphi(\langle a, b \rangle^*)$ if $H = (a, \dots, b)$. In Proposition 3.12 we shall see that

$$(3.7) \quad \lambda(\varphi; a, b) := \max\{\varphi(H) : H \text{ is a path from } a \text{ to } b\}$$

is optimal. It is easy to observe that $\lambda(\varphi; a, b) \geq \varphi(a \wedge b)$; in particular, $\lambda(\varphi; a, b) \geq 0$ if $\varphi \geq 0$. Furthermore, we can view $\lambda(\varphi; a, x)$ as a function of x , and obtain the monotonicity property.

Lemma 3.10. *If $\varphi \in M_1(L)$ then so does $\lambda(\varphi; a, \cdot)$.*

Proof. Let $H_1 = (a, \dots, x)$ be a path satisfying $\varphi(H_1) = \lambda(\varphi; a, x)$, and let $x < y$. Without loss of generality we can assume that $y \notin H_1$. Then we can add the edge $\{x, y\}$ to H_1 , and obtain the path $\tilde{H}_1 = (a, \dots, x, y)$. Since $\varphi(H_1) = \varphi(\tilde{H}_1) \leq \lambda(\varphi; a, y)$, we have shown that $\lambda(\varphi; a, \cdot)$ is monotone. \square

For any $a \in L$ we can introduce the λ -difference operator Λ_a by

$$(3.8) \quad \Lambda_a \varphi(x) = \varphi(x) - \lambda(\varphi; a, x), \quad x \in L.$$

By (3.6) and (3.7) we can easily see that (3.8) is expressed equivalently by

$$(3.9) \quad \Lambda_a \varphi(x) = \min\{\varphi(H; x) : H \text{ is a path from } a \text{ to } x\}.$$

Clearly $\Lambda_a \varphi \geq 0$ if φ is monotone, and it also possesses the monotonicity property.

Lemma 3.11. *If $\varphi \in M_1(L)$ then so does $\Lambda_a \varphi$.*

Proof. By (3.9) we can find a path $H_2 = (a, \dots, y)$ such that $\varphi(H_2; y) = \Lambda_a \varphi(y)$. Let $x < y$. If $x \in H_2$ then we can construct the path $\tilde{H}_2 = (a, \dots, x)$ by deleting all the edges from x to y in H_2 , and obtain $\varphi(H_2; y) \geq \varphi(\tilde{H}_2; x)$. Otherwise, we can add the edge $\{y, x\}$ to H_2 , and the resulting path $\tilde{H}_2 = (a, \dots, y, x)$ satisfies $\varphi(H_2; y) = \varphi(\tilde{H}_2; x)$. In either case we can show that $\varphi(H_2; y) \geq \varphi(\tilde{H}_2; x) \geq \Lambda_a \varphi(x)$. Therefore, $\Lambda_a \varphi$ is monotone. \square

Now we can prove the optimality of (3.7).

Proposition 3.12. $\lambda(\varphi; a, b) = B_\varphi(\langle a, b \rangle^*)$ for every pair $\{a, b\}$ of L .

Proof. For a fixed $a \in L$, we can decompose $\varphi(\cdot) = \lambda(\varphi; a, \cdot) + \Lambda_a \varphi(\cdot)$, in which $\lambda(\varphi; a, \cdot), \Lambda_a \varphi(\cdot) \in M_1(L)$ by Lemma 3.10 and 3.11. Thus, we can find completely monotone extensions Φ_1 and Φ_2 of $\lambda(\varphi; a, \cdot)$ and $\Lambda_a \varphi(\cdot)$ respectively, and construct $\Phi = \Phi_1 + \Phi_2$ so that $\Pi(\Phi) = \varphi$. Observe that

$$\Phi_2(\langle a, x \rangle^*) \leq \Phi_2(\langle a \rangle^*) = \Lambda_a \varphi(a) = 0.$$

and therefore, that

$$\Phi(\langle a, x \rangle^*) = \Phi_1(\langle a, x \rangle^*) \leq \Phi_1(\langle x \rangle^*) = \lambda(\varphi; a, x).$$

Since $\lambda(\varphi; a, x) \leq B_\varphi(\langle a, x \rangle^*)$ by Lemma 3.9, $\lambda(\varphi; a, x)$ attains $B_\varphi(\langle a, x \rangle^*)$. \square

3.2 Successive λ -difference operators

Given a sequence a_1, a_2, \dots from L , we can define the *successive λ -difference operator* recursively by

$$(3.10) \quad \Lambda_{a_1, \dots, a_n} \varphi = \Lambda_{a_n}(\Lambda_{a_1, \dots, a_{n-1}} \varphi), \quad n = 2, 3, \dots$$

The operator (3.8) maps from $M_1(L)$ to itself, and so does the operator (3.10). Unlike the operator (1.5), the definition of (3.10) depends on the order of a_i 's, as illustrated in the following example.

Example 3.13. Let $L = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234, 1234\}$ be a four-element Boolean lattice, and let

$$(3.11) \quad \varphi(x) = \begin{cases} 1 & \text{if } x = 1234; \\ 1/2 & \text{if } x = 123, 124 \text{ or } 234; \\ 1/3 & \text{if } x = 134, 13 \text{ or } 23; \\ 1/6 & \text{if } x = 12 \text{ or } 34; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\Lambda_{12,34} \varphi(234) = \frac{1}{3}$ and $\Lambda_{34,12} \varphi(234) = \frac{1}{6}$. If $x \neq 234$ then we obtain

$$\Lambda_{12,34} \varphi(x) = \Lambda_{34,12} \varphi(x) = \begin{cases} 2/3 & \text{if } x = 1234; \\ 1/3 & \text{if } x = 124; \\ 1/6 & \text{if } x = 13, 23, 123 \text{ or } 134; \\ 0 & \text{otherwise.} \end{cases}$$

We call a path (a_1, \dots, a_n) *monotone* if $i < j$ whenever $a_i < a_j$. As the following lemma suggests, we only need to consider a monotone path (a_1, \dots, a_n) for the operator $\Lambda_{a_1, \dots, a_n}$.

Lemma 3.14. *If $a_n \leq a_i$ for some $i \leq n-1$ then $\Lambda_{a_1, \dots, a_n} \varphi = \Lambda_{a_1, \dots, a_{n-1}} \varphi$ for every $\varphi \in M_1(L)$.*

Proof. Let $\varphi_{n-1} = \Lambda_{a_1, \dots, a_{n-1}} \varphi$. Since $a_n \leq a_i$, $\varphi_{n-1}(a_n) \leq \Lambda_{a_1, \dots, a_i} \varphi(a_n) = 0$. Thus, we can find that the path $H_0 = (a_n, a_n \wedge x, x)$ attains the minimum $\Lambda_{a_n} \varphi_{n-1}(x) = \varphi_{n-1}(x)$. \square

Here we set $\varphi_0 = \varphi \in M_1(L)$ and $\varphi_i = \Lambda_{a_i} \varphi_{i-1}$ recursively for $i = 1, \dots, n$. Then we can express φ_k by

$$(3.12) \quad \varphi_k(\cdot) = \sum_{i=k}^{n-1} \lambda(\varphi_i; a_{i+1}, \cdot) + \varphi_n(\cdot), \quad k = 0, \dots, n-1.$$

By choosing $\Psi_i \in \Pi^{-1}(\lambda(\varphi_i, a_{i+1}, \cdot))$ for $i = 0, \dots, n-1$, and $\Psi_n \in \Pi^{-1}(\varphi_n)$, we can construct

$$(3.13) \quad \Phi = \sum_{i=0}^n \Psi_i.$$

Comparing (3.13) with (3.12) at $k = 0$, we can easily observe that $\Pi(\Phi) = \varphi$. Theorem 3.15 characterizes $\Lambda_{a_1, \dots, a_k} \varphi$; in particular, when φ is a capacity there exists an \mathcal{L} -valued random variable \mathcal{X} satisfying (1.1) and (1.8).

Theorem 3.15. *Let (a_1, \dots, a_n) be a monotone path, and let*

$$(3.14) \quad \pi_{a_1, \dots, a_k}^x(V) = \begin{cases} 1 & \text{if } x \in V, a_i \notin V \text{ for all } i = 1, \dots, k; \\ 0 & \text{otherwise,} \end{cases}$$

be an indicator function on \mathcal{L} . Then (3.13) satisfies

$$(3.15) \quad \Lambda_{a_1, \dots, a_k} \varphi(x) = \Phi(\pi_{a_1, \dots, a_k}^x), \quad x \in L,$$

for $k = 1, \dots, n$.

Proof. Let f_i be the Möbius inverse of Ψ_i for $i = 0, \dots, n$. For each $i = 0, \dots, n-1$, note that $\lambda(\varphi_i; a_{i+1}, a_{i+1}) = \lambda(\varphi_i; a_{i+1}, \hat{1}) = \varphi_i(a_{i+1})$, and therefore, that $f_i(V) > 0$ implies $a_{i+1} \in V$. In particular, we find $\Psi_i(\pi_{a_1, \dots, a_k}^x) = 0$ for $i = 0, \dots, k-1$. For any $i = 1, \dots, k$ we can observe that $\lambda(\varphi_j; a_{j+1}, a_i) = 0$ for $j = k, \dots, n-1$ and that $\varphi_n(a_i) = 0$; thus, $f_j(V) = 0$ for $j = k, \dots, n$ if $a_i \in V$ for some $i = 1, \dots, k$. Hence, we obtain $\Psi_j(\pi_{a_1, \dots, a_k}^x) = \Psi_j(\langle x \rangle^*)$ for $j = k, \dots, n$. Together we can establish

$$\Phi(\pi_{a_1, \dots, a_k}^x) = \sum_{j=k}^n \Psi_j(\langle x \rangle^*) = \sum_{j=k}^{n-1} \lambda(\varphi_j; a_{j+1}, x) + \varphi_n(x) = \varphi_k(x)$$

where we can apply (3.12) for the last equality. \square

By Theorem 2.4 we can find that the operator ∇_{a_1, \dots, a_n} maps $M_\infty(L)$ to itself. Furthermore, it coincides with the operator $\Lambda_{a_1, \dots, a_n}$ on $M_\infty(L)$.

Lemma 3.16. $\Lambda_{a_1, \dots, a_n} \varphi = \nabla_{a_1, \dots, a_n} \varphi$ for $\varphi \in M_\infty(L)$.

Proof. We prove it by induction. Suppose that $\varphi_{n-1} = \Lambda_{a_1, \dots, a_{n-1}} \varphi = \nabla_{a_1, \dots, a_{n-1}} \varphi$. Since $\varphi_{n-1} \in M_\infty(L)$, by Propositions 3.5 and 3.7 we obtain $\lambda(\varphi_{n-1}; a_n, x) = \varphi_{n-1}(a_n \wedge x)$, and therefore, $\Lambda_{a_n} \varphi_{n-1} = \nabla_{a_n} \varphi_{n-1}$. \square

A monotone path (a_1, \dots, a_n) is viewed as a linear extension of L if $\{a_1, \dots, a_n\}$ is the entire set L . As a corollary to Lemma 3.16 we can find the uniqueness of (3.13) when $\varphi \in M_\infty(L)$.

Corollary 3.17. *If (a_1, \dots, a_n) is a linear extension of L and $\varphi \in M_\infty(L)$ then (3.13) is the Möbius extension of φ .*

Proof. Observe that $\varphi_n \equiv 0$, and that (3.13) becomes $\Phi = \sum_{i=0}^{n-1} \Psi_i$. As we have shown in the proof of Lemma 3.16, we have $\lambda(\varphi_i, a_{i+1}, x) = \varphi_i(a_{i+1} \wedge x)$ for $i = 0, \dots, n-1$. Since (a_1, \dots, a_n) is a linear extension of L , we can see that $\varphi_i(x) = 0$ if $x \leq a_i$, and therefore, that $\varphi_i(a_{i+1} \wedge x) = \varphi_i(a_{i+1})\chi_{\langle a_{i+1} \rangle^*}(x)$; thus, $\lambda(\varphi_i, a_{i+1}, \cdot)$ has the unique completely monotone extension $\Phi_i(V) = \varphi_i(a_{i+1})\chi_{\{\langle a_{i+1} \rangle^* \preceq V\}}$. Hence, Φ must be the Möbius extension of φ . \square

4 Probabilistic interpretation

By $C_1(L)$ we denote the collection of capacities on L , and by $C_\infty(\mathcal{L})$ the collection of completely monotone capacities on \mathcal{L} . Proposition 3.2 indicates that the projection Π is surjective from $C_\infty(\mathcal{L})$ onto $C_1(L)$. In view of (1.3) and Corollary 2.5 we can view any completely monotone capacity as a cdf. In this section we consider lattice-valued random variables on some probability space (Ω, \mathbb{P}) , and investigate their properties which facilitate a probabilistic interpretation of capacities.

4.1 Dual capacities

By L^* we denote the dual lattice of L , in which $\hat{0}$ and $\hat{1}$ respectively become the maximum and the minimum. Here we can introduce the successive difference operator ∇_{b_1, \dots, b_n} on L^* , and call it the *dual successive difference*, denoted by Δ_{b_1, \dots, b_n} . For any sequence b_1, b_2, \dots of L , it can be constructed with the dual difference operator

$$\Delta_{b_1}\varphi(x) = \varphi(x) - \varphi(x \vee b_1),$$

and recursively by

$$\Delta_{b_1, \dots, b_n}\varphi = \Delta_{b_n}(\Delta_{b_1, \dots, b_{n-1}}\varphi), \quad n = 2, 3, \dots$$

Then a capacity φ is called *completely alternating* if $\Delta_{b_1, \dots, b_n}\varphi \leq 0$ for any sequence b_1, \dots, b_n of L and for any $n \geq 1$. Given $\varphi \in C_1(L)$, we can introduce $\varphi^* \in C_1(L^*)$ by setting $\varphi^*(x) = 1 - \varphi(x)$ for $x \in L^*$, and call it the *dual capacity* of φ . The duality immediately implies that φ is completely alternating if and only if φ^* is completely monotone on L^* .

Let \mathcal{X} be an \mathcal{L} -valued random variable. Then $\varphi(x) = \mathbb{P}(x \in \mathcal{X})$ is a capacity if and only if $\mathbb{P}(\mathcal{X} = \langle \hat{0} \rangle^*) = 0$, in which $\Phi(U) = \mathbb{P}(\mathcal{X} \preceq U)$ is a completely monotone extension of φ . By \mathcal{L}^* we denote the distributive lattice of nonempty order ideals in L (i.e., the distributive lattice of nonempty dual order ideals in L^*) equipped with the reverse inclusion order \preceq (i.e., $D \preceq E$ on \mathcal{L}^* if $D \supseteq E$). Assume $\mathbb{P}(\mathcal{X} = \langle \hat{0} \rangle^*) = 0$. We can view the complement $\mathcal{X}^c = L \setminus \mathcal{X}$ as an \mathcal{L}^* -valued random variable, and define the *dual extension*

$$\Phi^*(D) = \mathbb{P}(\mathcal{X}^c \preceq D), \quad D \in \mathcal{L}^*.$$

It is easy to observe that

$$\mathbb{P}(x \in \mathcal{X}^c) = \mathbb{P}(x \notin \mathcal{X}) = 1 - \mathbb{P}(x \in \mathcal{X}) = 1 - \varphi(x) = \varphi^*(x),$$

and therefore, that Φ^* is a completely monotone extension of φ^* .

Suppose that φ is completely alternating and $\Phi^*(D) = \mathbb{P}(\mathcal{X}^c \preceq D)$ is the Möbius extension of φ^* . Then the *dual Möbius extension* $\Phi(U) = \mathbb{P}(\mathcal{X} \preceq U)$ has the Möbius inverse f supported by the collection

$$\{U \in \mathcal{L} : L \setminus U \text{ is a principal order ideal}\}.$$

Proposition 4.1. *A capacity φ is completely alternating and Φ is the dual Möbius extension of φ if and only if*

$$(4.1) \quad \Phi(\langle a, b \rangle^*) = \varphi(a) + \varphi(b) - \varphi(a \vee b) \text{ for every pair } \{a, b\}.$$

Proof. Let \mathcal{X} be an \mathcal{L} -valued random variable realizing $\varphi(x) = \mathbb{P}(x \in \mathcal{X})$. Then $\mathcal{X}^c = L \setminus \mathcal{X}$ realizes its dual $\varphi^*(x) = \mathbb{P}(x \in \mathcal{X}^c)$. Thus, we obtain

$$\begin{aligned} \Phi(\langle a, b \rangle^*) &= \mathbb{P}(\mathcal{X} \preceq \langle a, b \rangle^*) = \mathbb{P}(a \notin \mathcal{X}^c, b \notin \mathcal{X}^c) \\ &= 1 - \mathbb{P}(a \in \mathcal{X}^c) - \mathbb{P}(b \in \mathcal{X}^c) + \mathbb{P}(a, b \in \mathcal{X}^c) \\ &= \varphi(a) + \varphi(b) - \varphi(a \vee b) + \mathbb{P}(a, b \in \mathcal{X}^c, a \vee b \notin \mathcal{X}^c). \end{aligned}$$

If Φ is the dual Möbius extension of φ then $\mathbb{P}(a, b \in \mathcal{X}^c, a \vee b \notin \mathcal{X}^c) = 0$. Conversely if (4.1) holds then Φ^* must be the Möbius extension of φ^* . \square

Since $\varphi(H) = \varphi(a) + \varphi(b) - \varphi(a \vee b)$ for a path $H = (a, b)$, the dual Möbius extension $\Phi(\langle a, b \rangle^*)$ in (4.1) attains the Fréchet bound $B_\varphi(\langle a, b \rangle^*)$.

4.2 Stochastic inequalities

When $\varphi \in C_\infty(L)$ is a cdf for L -valued random variable X , by Theorem 2.4 we can show that

$$(4.2) \quad \mathbb{P}(X \notin \langle A \rangle) = \nabla_A^1 \varphi, \quad A \subseteq L.$$

Suppose that (X, Y) is a pair of L -valued random variables. We can construct such a pair satisfying $\mathbb{P}(X \leq Y) = 1$ if and only if

$$(4.3) \quad \mathbb{P}(X \in U) \leq \mathbb{P}(Y \in U) \quad \text{for every } U \in \mathcal{L},$$

given the marginal conditions $\varphi(x) = \mathbb{P}(X \leq x)$ and $\psi(y) = \mathbb{P}(Y \leq y)$. By applying (4.2), we can immediately observe that (4.3) can be equivalently stated by

$$(4.4) \quad \nabla_{a_1, \dots, a_k} \varphi(\hat{1}) \leq \nabla_{a_1, \dots, a_k} \psi(\hat{1}) \quad \text{for every antichain } \{a_1, \dots, a_k\} \text{ in } L.$$

The stochastic inequality (4.3) first appeared in the paper by Kamae, Krengel, and O'Brien [3], and (4.4) was shown by Norberg [8] in the context of random sets.

Let \mathcal{X} be an \mathcal{L} -valued random variable, and let Y be an L -valued random variable. In this subsection we will investigate when we can construct a pair (\mathcal{X}, Y) of random variables jointly so that $\mathbb{P}(Y \in \mathcal{X}) = 1$ given the marginal conditions

$$(4.5) \quad \varphi(x) = \mathbb{P}(x \in \mathcal{X}) \text{ and } \psi(y) = \mathbb{P}(Y \leq y), \quad x, y \in L.$$

The joint cdf $\Gamma(V, y) = \mathbb{P}(\mathcal{X} \preceq V, Y \leq y)$ is a completely monotone capacity on the direct product lattice $\mathcal{L} \times L$. Given a joint cdf Γ , we can introduce the expectation $E[w(\mathcal{X}, Y)]$, also denoted by $\Gamma(w)$, for $w \in R(\mathcal{L} \times L)$. Then we can define the Fréchet bound

$$B^{(\varphi, \psi)}(w) = \max\{\Gamma(w) \text{ subject to (4.5)}\}, \quad w \in R(\mathcal{L} \times L).$$

Similarly by $\psi(h)$ we denote the expectation $E[h(Y)]$ for $h \in R(L)$. Recall the dual problem $S^\varphi(g)$ in Theorem 3.6. In Theorem 4.2 we will show that the Fréchet bound $B^{(\varphi, \psi)}(w)$ has the dual problem

$$(4.6) \quad S_{(\varphi, \psi)}(w) = \min\{\psi(h) - S^\varphi(g) \text{ subject to (4.7)}\}$$

with the inequality constraint

$$(4.7) \quad w(V, y) \leq h(y) - g(V), \quad (V, y) \in \mathcal{L} \times L,$$

for $(g, h) \in R(\mathcal{L}) \times R(L)$.

Theorem 4.2. $B^{(\varphi, \psi)}(w) = S_{(\varphi, \psi)}(w)$ for any $w \in R(\mathcal{L} \times L)$.

Proof. Suppose that a joint cdf Γ for (\mathcal{X}, Y) attains $B^{(\varphi, \psi)}(w)$, that (g, h) attains $S_{(\varphi, \psi)}(w)$, and that r is of the form (3.5) satisfying $r \leq g$ and $S^\varphi(g) = E[r(\mathcal{X})]$. Then we can observe that

$$S_{(\varphi, \psi)}(w) = \psi(h) - S^\varphi(g) = E[h(Y)] - E[r(\mathcal{X})] \geq E[w(\mathcal{X}, Y)] = B^{(\varphi, \psi)}(w),$$

and that the equality holds if $w(V, y) = h(y) - r(V)$, $(V, y) \in \mathcal{L} \times L$. Since $S_{(\varphi, \psi)}(w_1 + w_2) \leq S_{(\varphi, \psi)}(w_1) + S_{(\varphi, \psi)}(w_2)$, we can apply the Hahn-Banach theorem analogous to the proof of Theorem 3.6, and conclude that $B^{(\varphi, \psi)}(w) = S_{(\varphi, \psi)}(w)$. \square

In what follows we consider the indicator function $w_1(V, y) := \chi_{\{y \in V\}}$ for the dual problem (4.6). Starting with $g \in R(\mathcal{L})$, we can construct the two monotone functions h' and g' by

$$(4.8) \quad h'(y) = \max_{V \in \mathcal{L}} (w_1(V, y) + g(V)), \quad y \in L;$$

$$(4.9) \quad g'(V) = \min_{y \in L} (h'(y) - w_1(V, y)), \quad V \in \mathcal{L}.$$

By (4.9) we can see that (4.7) holds for w_1 , h' , and g' . Observe that if w_1 , h , and g satisfy (4.7) then $h \geq h'$ and $g' \geq g$ so that $\psi(h) - S^\varphi(g) \geq \psi(h') - S^\varphi(g')$. Thus, it suffices for us to consider the case when g and h are monotone. Moreover, without loss of generality we can set $g(\langle \hat{1} \rangle^*) = 0$ in addition to the constraint (4.7). Starting with a monotone function g with $g(\langle \hat{1} \rangle^*) = 0$, we obtain $0 \leq h'(y) \leq 1$ in (4.8), and $g'(V) = \min_{y \in V} h'(y) - 1$ in (4.9). Therefore, we can further simplify (4.6) into

$$(4.10) \quad S_{(\varphi, \psi)}(w_1) = \min\{\psi(h) - S^\varphi(\tilde{h}) \text{ subject to (4.11)}\} + 1$$

with the constraint

$$(4.11) \quad 0 \leq h(y) \leq 1 \text{ and } \tilde{h}(V) = \min_{y \in V} h(y), \quad (V, y) \in \mathcal{L} \times L,$$

for any monotone function $h \in R(L)$.

Theorem 4.3. *If*

$$(4.12) \quad \Lambda_{a_1, \dots, a_k} \varphi(\hat{1}) \leq \nabla_{a_1, \dots, a_k} \psi(\hat{1}) \quad \text{for every monotone path } (a_1, \dots, a_k),$$

then there exists a joint cdf Γ for (\mathcal{X}, Y) satisfying $\mathbb{P}(Y \in \mathcal{X}) = 1$ given the marginal conditions (4.5).

Proof. Suppose that h is a monotone function, and that (4.11) holds for (h, \tilde{h}) . Then we can find a linear extension (a_1, \dots, a_N) of L such that $h(a_i) \leq h(a_j)$ whenever $i < j$. By Theorem 3.15 we can construct $\Phi \in \Pi^{-1}(\varphi)$ so that (3.15) holds for the indicator function π_{a_1, \dots, a_k}^x with any choice of $k = 1, \dots, N$. For each $0 \leq t < h(\hat{1})$, note that there is some $k \leq N - 1$ such that

$$\begin{aligned} A(t) &:= \{y \in L : h(y) > t\} = \{a_{k+1}, \dots, a_N\}; \\ \mathcal{A}(t) &:= \{V \in \mathcal{L} : \tilde{h}(V) > t\} = \{V \in \mathcal{L} : a_i \notin V, i = 1, \dots, k\}. \end{aligned}$$

By applying Theorems 2.4 and 3.15, we can establish

$$\begin{aligned} \psi(h) - S^\varphi(\tilde{h}) &\geq \psi(h) - \Phi(\tilde{h}) = \int_0^{h(\hat{1})} \psi(\chi_{A(t)}) dt - \int_0^{h(\hat{1})} \Phi(\chi_{\mathcal{A}(t)}) dt \\ &\geq \min_{1 \leq k \leq N-1} \left[\psi(\chi_{\{a_{k+1}, \dots, a_N\}}) - \Phi(\pi_{a_1, \dots, a_k}^{\hat{1}}) \right] \\ &= \min_{1 \leq k \leq N-1} [\nabla_{a_1, \dots, a_k} \psi(\hat{1}) - \Lambda_{a_1, \dots, a_k} \varphi(\hat{1})] \geq 0. \end{aligned}$$

By Theorem 4.2 and (4.10) we obtain $B^{(\varphi, \psi)}(w_1) = S_{(\varphi, \psi)}(w_1) \geq 1$, which implies the existence of a joint cdf Γ satisfying $\mathbb{P}(Y \in \mathcal{X}) = 1$. \square

Example 4.4. A stochastically comparable pair (\mathcal{X}, Y) does not necessarily satisfy (4.12). Let L be the Boolean lattice from Example 3.13, and let

$$\Gamma(V, y) = \begin{cases} 1/6 & \text{if } (V, y) = (\langle 12 \rangle^*, 12), (\langle 13, 23, 34 \rangle^*, 34), (\langle 13, 23 \rangle^*, 234) \\ & \text{or } (\langle 234 \rangle^*, 234); \\ 1/3 & \text{if } (V, y) = (\langle 124 \rangle^*, 124); \\ 0 & \text{otherwise,} \end{cases}$$

be a joint cdf for (\mathcal{X}, Y) . Then it satisfies $\mathbb{P}(Y \in \mathcal{X}) = 1$, and $\varphi(x) = \mathbb{P}(x \in \mathcal{X})$ is equal to (3.11). By applying the result of Example 3.13, we can calculate $\Lambda_{34, 12, 234} \varphi(\hat{1}) = \Lambda_{34, 12} \varphi(\hat{1}) - \Lambda_{34, 12} \varphi(234) = 1/2$. Since $\nabla_{34, 12, 234} \psi(\hat{1}) = \mathbb{P}(Y \notin \langle 12, 234 \rangle) = 1/3$, it does not satisfy (4.12) for the monotone path $(34, 12, 234)$.

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